Supplemental Information: Multi-time Formulation of Matsubara Dynamics

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I. COMPLETE DERIVATION OF EQ. (9) IN THE MAIN TEXT

Beginning with the Kubo-transformed time correlation function

$$K_{AB}(t) = \frac{1}{Z\beta}\int_0^\beta d\lambda\, Tr\left[e^{-(\beta-\lambda)\hat{H}}\hat{A}e^{-\lambda\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right], \tag{1}$$

where β is the inverse temperature and Z the partition function, the integral over lambda can be discretized as

$$\begin{aligned}
K_{AB}^{[N]}(t) &= \frac{1}{Z_N N}\sum_{k=1}^{N} Tr\left[e^{-\beta_N(N-k)\hat{H}}\hat{A}e^{-\beta_N k\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right] \\
&= \frac{1}{Z_N N}\sum_{k=1}^{N} Tr\left[\left(e^{-\beta_N\hat{H}}\right)^{N-k}\hat{A}\left(e^{-\beta_N\hat{H}}\right)^{k}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right] \\
&= \frac{1}{Z_N N}\sum_{k=1}^{N} Tr\left[\left(e^{-\beta_N\hat{H}}\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}\left(e^{-\beta_N\hat{H}}\right)^{k-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right],
\end{aligned} \tag{2}$$

where $\beta_N = \beta/N$ and $Z_N = Tr\left[\left(e^{-\beta_N\hat{H}}\right)^N\right]$. We will now insert $N-1$ identities of the form

$$\hat{1}_t = e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}, \tag{3}$$

in Eq. (2) to give:

$$\begin{aligned}
K_{AB}^{[N]}(t) &= \frac{1}{Z_N N}\sum_{k=1}^{N} Tr\left[\left(e^{-\beta_N\hat{H}}\hat{1}_t\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}\hat{1}_t\left(e^{-\beta_N\hat{H}}\hat{1}_t\right)^{k-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right] \\
&= \frac{1}{Z_N N}\sum_{k=1}^{N} Tr\Bigg[\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar} \\
&\quad\times\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}\right)^{k-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\Bigg],
\end{aligned} \tag{4}$$

which is Eq. (4) in the main text. The trace will then be expanded in the position basis to give:

$$\begin{aligned}
K_{AB}^{[N]}(t) &= \frac{1}{Z_N N}\sum_{k=1}^{N}\int dq_N''\langle q_N''|\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar} \\
&\quad\times\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}\right)^{k-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}|q_N''\rangle.
\end{aligned} \tag{5}$$

Now consider the jth term in the sum of Eq. (5):

$$
\begin{aligned}
&\int dq''_N \langle q''_N| \left(e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar}\right)^{N-j-1} e^{-\beta_N \hat{H}} \hat{A} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar} \\
&\times \left(e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar}\right)^{j-1} e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} \hat{B} e^{-i\hat{H}t/\hbar} |q''_N\rangle \\
= &\int dq''_N \langle q''_N| \underbrace{e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar} \cdots e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar}}_{N-j-1 \text{ terms}} e^{-\beta_N \hat{H}} \hat{A} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar} \\
&\times \underbrace{e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar} \cdots e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} e^{-i\hat{H}t/\hbar}}_{j-1 \text{ terms}} e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar} \hat{B} e^{-i\hat{H}t/\hbar} |q''_N\rangle.
\end{aligned}
\tag{6}
$$

We will insert identities of the form

$$
\hat{1}_{q'_l} = \int dq'_l |q'_l\rangle\langle q'_l| \tag{7}
$$

after each $e^{-\beta_N \hat{H}}$ term,

$$
\hat{1}_{z_l} = \int dz_l |z_l\rangle\langle z_l| \tag{8}
$$

after each $e^{i\hat{H}t/\hbar}$ term and

$$
\hat{1}_{q''_l} = \int dq''_l |q''_l\rangle\langle q''_l| \tag{9}
$$

after each $e^{-i\hat{H}t/\hbar}$ term (except the one next to $|q''_N\rangle$ since the expansion of the trace takes care of this term) to get

$$
\begin{aligned}
= &\int dq''_N \langle q''_N| e^{-\beta_N \hat{H}} \hat{1}_{q'_1} e^{i\hat{H}t/\hbar} \hat{1}_{z_1} e^{-i\hat{H}t/\hbar} \hat{1}_{q''_1} \cdots e^{-\beta_N \hat{H}} \hat{1}_{q'_{N-j-1}} e^{i\hat{H}t/\hbar} \hat{1}_{z_{N-j-1}} e^{-i\hat{H}t/\hbar} \hat{1}_{q''_{N-j-1}} \\
&\times e^{-\beta_N \hat{H}} \hat{1}_{q'_{N-j}} \hat{A} e^{i\hat{H}t/\hbar} \hat{1}_{z_{N-j}} e^{-i\hat{H}t/\hbar} \hat{1}_{q''_{N-j}} e^{-\beta_N \hat{H}} \hat{1}_{q'_{N-j+1}} e^{i\hat{H}t/\hbar} \hat{1}_{z_{N-j+1}} e^{-i\hat{H}t/\hbar} \hat{1}_{q''_{N-j+1}} \\
&\times \cdots e^{-\beta_N \hat{H}} \hat{1}_{q'_{N-1}} e^{i\hat{H}t/\hbar} \hat{1}_{z_{N-1}} e^{-i\hat{H}t/\hbar} \hat{1}_{q''_{N-1}} e^{-\beta_N \hat{H}} \hat{1}_{q'_N} e^{i\hat{H}t/\hbar} \hat{1}_{z_N} \hat{B} e^{-i\hat{H}t/\hbar} |q''_N\rangle \\
= &\int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' \langle q''_N| e^{-\beta_N \hat{H}} |q'_1\rangle \langle q'_1| e^{i\hat{H}t/\hbar} |z_1\rangle \langle z_1| e^{-i\hat{H}t/\hbar} |q''_1\rangle \cdots \langle q''_{N-j-2}| e^{-\beta_N \hat{H}} |q'_{N-j-1}\rangle \\
&\times \langle q'_{N-j-1}| e^{i\hat{H}t/\hbar} |z_{N-j-1}\rangle \langle z_{N-j-1}| e^{-i\hat{H}t/\hbar} |q''_{N-j-1}\rangle \langle q''_{N-j-1}| e^{-\beta_N \hat{H}} |q'_{N-j}\rangle \\
&\times \langle q'_{N-j}| \hat{A} e^{i\hat{H}t/\hbar} |z_{N-j}\rangle \langle z_{N-j}| e^{-i\hat{H}t/\hbar} |q''_{N-j}\rangle \langle q''_{N-j}| e^{-\beta_N \hat{H}} |q'_{N-j+1}\rangle \langle q'_{N-j+1}| e^{i\hat{H}t/\hbar} |z_{N-j+1}\rangle \\
&\times \langle z_{N-j+1}| e^{-i\hat{H}t/\hbar} |q''_{N-j+1}\rangle \cdots \langle q''_{N-2}| e^{-\beta_N \hat{H}} |q'_{N-1}\rangle \langle q'_{N-1}| e^{i\hat{H}t/\hbar} |z_{N-1}\rangle \langle z_{N-1}| e^{-i\hat{H}t/\hbar} |q''_{N-1}\rangle \\
&\times \langle q''_{N-1}| e^{-\beta_N \hat{H}} |q'_N\rangle \langle q'_N| e^{i\hat{H}t/\hbar} |z_N\rangle \langle z_N| \hat{B} e^{-i\hat{H}t/\hbar} |q''_N\rangle,
\end{aligned}
$$

where we introduced the notation (for $x = \{q', q'', z\}$)

$$
\int d\boldsymbol{x} = \prod_{k=1}^{N} \int dx_k. \tag{10}
$$

Assuming that $\hat{A}$ and $\hat{B}$ are functions of the position operator, the previous equation can be expressed as

$$
\begin{aligned}
&= \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' A(q'_{N-j}) B(z_N) \langle q''_N | e^{-\beta_N \hat{H}} | q'_1 \rangle \langle q'_1 | e^{i\hat{H}t/\hbar} | z_1 \rangle \langle z_1 | e^{-i\hat{H}t/\hbar} | q''_1 \rangle \cdots \langle q''_{N-j-2} | e^{-\beta_N \hat{H}} | q'_{N-j-1} \rangle \\
&\quad \times \langle q'_{N-j-1} | e^{i\hat{H}t/\hbar} | z_{N-j-1} \rangle \langle z_{N-j-1} | e^{-i\hat{H}t/\hbar} | q''_{N-j-1} \rangle \langle q''_{N-j-1} | e^{-\beta_N \hat{H}} | q'_{N-j} \rangle \\
&\quad \times \langle q'_{N-j} | e^{i\hat{H}t/\hbar} | z_{N-j} \rangle \langle z_{N-j} | e^{-i\hat{H}t/\hbar} | q''_{N-j} \rangle \langle q''_{N-j} | e^{-\beta_N \hat{H}} | q'_{N-j+1} \rangle \langle q'_{N-j+1} | e^{i\hat{H}t/\hbar} | z_{N-j+1} \rangle \\
&\quad \times \langle z_{N-j+1} | e^{-i\hat{H}t/\hbar} | q''_{N-j+1} \rangle \cdots \langle q''_{N-2} | e^{-\beta_N \hat{H}} | q'_{N-1} \rangle \langle q'_{N-1} | e^{i\hat{H}t/\hbar} | z_{N-1} \rangle \langle z_{N-1} | e^{-i\hat{H}t/\hbar} | q''_{N-1} \rangle \\
&\quad \times \langle q''_{N-1} | e^{-\beta_N \hat{H}} | q'_N \rangle \langle q'_N | e^{i\hat{H}t/\hbar} | z_N \rangle \langle z_N | e^{-i\hat{H}t/\hbar} | q''_N \rangle. \\
&= \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' A(q'_{N-j}) B(z_N) \prod_{l=1}^{N} \langle q''_{l-1} | e^{-\beta_N \hat{H}} | q'_l \rangle \langle q'_l | e^{i\hat{H}t/\hbar} | z_l \rangle \langle z_l | e^{-i\hat{H}t/\hbar} | q''_l \rangle. \qquad (11)
\end{aligned}
$$

where we have used the cyclic condition $x_0 = x_N$.

Going back to the sum in Eq. (5), and using the result of Eq. (11), it follows that

$$
\begin{aligned}
K^{[N]}_{AB}(t) &= \frac{1}{Z_N N} \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' \sum_{k=1}^{N} A(q'_{N-k}) B(z_N) \prod_{l=1}^{N} \langle q''_{l-1} | e^{-\beta_N \hat{H}} | q'_l \rangle \langle q'_l | e^{i\hat{H}t/\hbar} | z_l \rangle \langle z_l | e^{-i\hat{H}t/\hbar} | q''_l \rangle \\
&= \frac{1}{Z_N N} \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' \sum_{k=1}^{N} A(q'_k) B(z_N) \prod_{l=1}^{N} \langle q''_{l-1} | e^{-\beta_N \hat{H}} | q'_l \rangle \langle q'_l | e^{i\hat{H}t/\hbar} | z_l \rangle \langle z_l | e^{-i\hat{H}t/\hbar} | q''_l \rangle \\
&= \frac{1}{Z_N} \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' A(\boldsymbol{q}') B(z_N) \prod_{l=1}^{N} \langle q''_{l-1} | e^{-\beta_N \hat{H}} | q'_l \rangle \langle q'_l | e^{i\hat{H}t/\hbar} | z_l \rangle \langle z_l | e^{-i\hat{H}t/\hbar} | q''_l \rangle \\
&= \frac{1}{Z_N} \int d\boldsymbol{q}'' \int d\boldsymbol{z} \int d\boldsymbol{q}' A(\boldsymbol{q}') B(\boldsymbol{z}) \prod_{l=1}^{N} \langle q''_{l-1} | e^{-\beta_N \hat{H}} | q'_l \rangle \langle q'_l | e^{i\hat{H}t/\hbar} | z_l \rangle \langle z_l | e^{-i\hat{H}t/\hbar} | q''_l \rangle, \qquad (12)
\end{aligned}
$$

where

$$O(\boldsymbol{x}) = \frac{1}{N} \sum_{k=1}^{N} O(x_k), \qquad (13)$$

is the block-averaged observable. Note that in the last step we made use of the equivalence of $\{z_j\}$ points in the (cyclic) path integral representation and average the observable $B(z_N)$ over all **z** coordinates. Eq. (12) corresponds to Eq. (9) in the main text.

II. COMPLETE DERIVATION OF EQ. (55) IN THE MAIN TEXT

Let's begin with the symmetrized double Kubo transform defined as (see Eq. 48 in the main text)

$$K^{sym}_{ABC}(t, t') = I_1(t, t') + I_2(t, t') \qquad (14)$$

where

$$I_1(t,t') = \frac{1}{Z\beta^2}\int_0^\beta d\lambda \int_0^\lambda d\lambda' Tr\left[e^{-(\beta-\lambda)\hat{H}}\hat{A}e^{-(\lambda-\lambda')\hat{H}}\hat{B}(t)e^{-\lambda'\hat{H}}\hat{C}(t')\right] \tag{15}$$

and

$$I_2(t,t') = \frac{1}{Z\beta^2}\int_0^\beta d\lambda \int_0^\lambda d\lambda' Tr\left[e^{-(\beta-\lambda)\hat{H}}\hat{B}(t)e^{-(\lambda-\lambda')\hat{H}}\hat{A}e^{-\lambda'\hat{H}}\hat{C}(t')\right]. \tag{16}$$

Focusing only in the $I_1(t,t')$ term, the double integral over lambda can be discretized as

$$\begin{aligned} I_1(t,t') &= \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N\hat{H}}\right)^{N-k}\hat{A}\left(e^{-\beta_N\hat{H}}\right)^{k-l}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\left(e^{-\beta_N\hat{H}}\right)^{l}e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right] \\ &= \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N\hat{H}}\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}\left(e^{-\beta_N\hat{H}}\right)^{k-l-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\right. \\ &\quad \left.\times\left(e^{-\beta_N\hat{H}}\right)^{l-1}e^{-\beta_N\hat{H}}e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right]. \end{aligned} \tag{17}$$

We will now insert $N-1$ identities of the form

$$\hat{1}_t = e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar} \tag{18}$$

and

$$\hat{1}_{t'} = e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}, \tag{19}$$

in Eq. (17) to give:

$$\begin{aligned} I_1(t,t') &= \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N\hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{N-k-1}e^{-\beta_N\hat{H}}\hat{A}\hat{1}_t\hat{1}_{t'}\left(e^{-\beta_N\hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{k-l-1}\right. \\ &\quad \left.\times e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\hat{1}_{t'}\left(e^{-\beta_N\hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{l-1}e^{-\beta_N\hat{H}}\hat{1}_t e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right] \\ &= \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{N-k-1}\right. \\ &\quad \times e^{-\beta_N\hat{H}}\hat{A}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{k-l-1} \\ &\quad \times e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\left(e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{l-1} \\ &\quad \left.\times e^{-\beta_N\hat{H}}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right], \end{aligned} \tag{20}$$

which is Eq. (52) in the main text.

Following the steps that lead to the discretization of the single time Kubo-transformed correlation function in the previous section, one can expand the trace in a position basis and insert the identities introduced in Eqs. (7)-(9) along with the additional identity

$$\hat{1}_{z'_l} = \int dz'_l |z'_l\rangle\langle z'_l|, \tag{21}$$

between each $e^{i\hat{H}t'/\hbar}$ and $e^{-i\hat{H}t'/\hbar}$ to obtain

$$\begin{aligned}
I_1(t,t') = &\frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k}\int dq''_N \,\langle q''_N|\, e^{-\beta_N\hat{H}}\hat{1}_{q'_1}e^{i\hat{H}t/\hbar}\hat{1}_{z_1}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{1}_{z'_1}e^{-i\hat{H}t'/\hbar}\hat{1}_{q''_1}\cdots \\
&\times e^{-\beta_N\hat{H}}\hat{1}_{q'_{N-k-1}}e^{i\hat{H}t/\hbar}\hat{1}_{z_{N-k-1}}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{1}_{z'_{N-k-1}}e^{-i\hat{H}t'/\hbar}\hat{1}_{q''_{N-k-1}} \\
&\times e^{-\beta_N\hat{H}}\hat{1}_{q'_{N-k}}\hat{A}e^{i\hat{H}t/\hbar}\hat{1}_{z_{N-k}}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{1}_{z'_{N-k}}e^{-i\hat{H}t'/\hbar}\hat{1}_{q''_{N-k}}\cdots \\
&\times e^{-\beta_N\hat{H}}\hat{1}_{q'_{k-l-1}}e^{i\hat{H}t/\hbar}\hat{1}_{z_{k-l-1}}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{1}_{z'_{k-l-1}}e^{-i\hat{H}t'/\hbar}\hat{1}_{q''_{k-l-1}} \\
&\times e^{-\beta_N\hat{H}}\hat{1}_{q'_{k-l}}e^{i\hat{H}t/\hbar}\hat{B}\hat{1}_{z_{k-l}}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{1}_{z'_{k-l}}e^{-i\hat{H}t'/\hbar}\hat{1}_{q''_{k-l}}\cdots \\
&\times e^{-\beta_N\hat{H}}\hat{1}_{q'_N}e^{i\hat{H}t/\hbar}\hat{1}_{z_N}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{C}\hat{1}_{z'_N}e^{-i\hat{H}t'/\hbar}|q''_N\rangle \\
= &\frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k}\int d\boldsymbol{q}''\int d\boldsymbol{z}\int d\boldsymbol{q}'\int d\boldsymbol{z'}\,A(q'_{N-k})B(z_{k-l})C(z'_N) \\
&\times\langle q''_N|e^{-\beta_N\hat{H}}|q'_1\rangle\langle q'_1|e^{i\hat{H}t/\hbar}|z_1\rangle\langle z_1|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_1\rangle\langle z'_1|e^{-i\hat{H}t'/\hbar}|q''_1\rangle\cdots \\
&\times\langle q''_{N-k-1}|e^{-\beta_N\hat{H}}|q'_{N-k}\rangle\langle q'_{N-k}|e^{i\hat{H}t/\hbar}|z_{N-k}\rangle\langle z_{N-k}|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_{N-k}\rangle\langle z'_{N-k}|e^{-i\hat{H}t'/\hbar}|q''_{N-k}\rangle\cdots \\
&\times\langle q''_{k-l-1}|e^{-\beta_N\hat{H}}|q'_{k-l}\rangle\langle q'_{k-l}|e^{i\hat{H}t/\hbar}|z_{k-l}\rangle\langle z_{k-l}|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_{k-l}\rangle\langle z'_{k-l}|e^{-i\hat{H}t'/\hbar}|q''_{k-l}\rangle\cdots \\
&\times\langle q''_{N-1}|e^{-\beta_N\hat{H}}|q'_N\rangle\langle q'_N|e^{i\hat{H}t/\hbar}|z_N\rangle\langle z_N|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_N\rangle\langle z'_N|e^{-i\hat{H}t'/\hbar}|q''_N\rangle \\
= &\frac{1}{Z_N N^2}\int d\boldsymbol{q}''\int d\boldsymbol{z}\int d\boldsymbol{q}'\int d\boldsymbol{z'}\sum_{k=1}^{N}A(q'_{N-k})\sum_{l=1}^{k}B(z_{k-l})C(z'_N) \\
&\times\prod_{l=1}^{N}\langle q''_{l-1}|e^{-\beta_N\hat{H}}|q'_l\rangle\langle q'_l|e^{i\hat{H}t/\hbar}|z_l\rangle\langle z_l|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_l\rangle\langle z'_l|e^{-i\hat{H}t'/\hbar}|q''_l\rangle.
\end{aligned} \tag{22}$$

where we have used the fact that $\hat{A}$, $\hat{B}$ and $\hat{C}$ are functions of position. Take note that no identity operators are placed in between the real t and t' propagators.

Following similar steps, the integral over lambda in the $I_2(t,t')$ term of Eq. (14) can be discretized as

$$I_2(t,t') = \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N \hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{N-k-1} e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}\hat{1}_{t'}\left(e^{-\beta_N \hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{k-l-1}\right.$$
$$\left.\times e^{-\beta_N \hat{H}}\hat{A}\hat{1}_t\hat{1}_{t'}\left(e^{-\beta_N \hat{H}}\hat{1}_t\hat{1}_{t'}\right)^{l-1} e^{-\beta_N \hat{H}}\hat{1}_t e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right]$$
$$= \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k} Tr\left[\left(e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{N-k-1}\right.$$
$$\times e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}\hat{B}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\left(e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{k-l-1}$$
$$\times e^{-\beta_N \hat{H}}\hat{A}e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\left(e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}e^{-i\hat{H}t'/\hbar}\right)^{l-1}$$
$$\left.\times e^{-\beta_N \hat{H}} e^{i\hat{H}t/\hbar}e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}\hat{C}e^{-i\hat{H}t'/\hbar}\right], \tag{23}$$

which is Eq. (53) in the main text. Expanding the trace in a position basis and inserting the identities of Eqs. (7)-(9) and (21), the path integral discretization of $I_2(t,t')$ can be cast as

$$I_2(t,t') = \frac{1}{Z_N N^2}\sum_{k=1}^{N}\sum_{l=1}^{k}\int d\boldsymbol{q}''\int d\boldsymbol{z}\int d\boldsymbol{q}'\int d\boldsymbol{z'}B(z_{N-k})A(q'_{k-l})C(z'_N)$$
$$\times\langle q''_N|e^{-\beta_N\hat{H}}|q'_1\rangle\langle q'_1|e^{i\hat{H}t/\hbar}|z_1\rangle\langle z_1|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_1\rangle\langle z'_1|e^{-i\hat{H}t'/\hbar}|q''_1\rangle\cdots$$
$$\times\langle q''_{N-k-1}|e^{-\beta_N\hat{H}}|q'_{N-k}\rangle\langle q'_{N-k}|e^{i\hat{H}t/\hbar}|z_{N-k}\rangle\langle z_{N-k}|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_{N-k}\rangle\langle z'_{N-k}|e^{-i\hat{H}t'/\hbar}|q''_{N-k}\rangle\cdots$$
$$\times\langle q''_{k-l-1}|e^{-\beta_N\hat{H}}|q'_{k-l}\rangle\langle q'_{k-l}|e^{i\hat{H}t/\hbar}|z_{k-l}\rangle\langle z_{k-l}|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_{k-l}\rangle\langle z'_{k-l}|e^{-i\hat{H}t'/\hbar}|q''_{k-l}\rangle\cdots$$
$$\times\langle q''_{N-1}|e^{-\beta_N\hat{H}}|q'_N\rangle\langle q'_N|e^{i\hat{H}t/\hbar}|z_N\rangle\langle z_N|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_N\rangle\langle z'_N|e^{-i\hat{H}t'/\hbar}|q''_N\rangle$$
$$= \frac{1}{Z_N N^2}\int d\boldsymbol{q}''\int d\boldsymbol{z}\int d\boldsymbol{q}'\int d\boldsymbol{z'}\sum_{k}^{N}B(z_{N-k})\sum_{l}^{k}A(q_{k-l})C(z'_N)$$
$$\times\prod_{l=1}^{N}\langle q''_{l-1}|e^{-\beta_N\hat{H}}|q'_l\rangle\langle q'_l|e^{i\hat{H}t/\hbar}|z_l\rangle\langle z_l|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_l\rangle\langle z'_l|e^{-i\hat{H}t'/\hbar}|q''_l\rangle. \tag{24}$$

Adding Eq. (22) and Eq. (24) and using the identity

$$\sum_{i=1}^{N}\sum_{j=1}^{i}\left[A(x_i)B(y_j)+B(y_i)A(x_j)\right] = \sum_{i=1}^{N}\sum_{j=1}^{N}A(x_i)B(y_j) \tag{25}$$

the symmetrized double Kubo transform can be expressed as

$$K^{sym}_{ABC}(t,t') = \frac{1}{Z_N}\int d\boldsymbol{q}''\int d\boldsymbol{z}\int d\boldsymbol{q}'\int d\boldsymbol{z'}A(\boldsymbol{q})B(\boldsymbol{z})C(\boldsymbol{z'})$$
$$\times\prod_{l=1}^{N}\langle q''_{l-1}|e^{-\beta_N\hat{H}}|q'_l\rangle\langle q'_l|e^{i\hat{H}t/\hbar}|z_l\rangle\langle z_l|e^{-i\hat{H}t/\hbar}e^{i\hat{H}t'/\hbar}|z'_l\rangle\langle z'_l|e^{-i\hat{H}t'/\hbar}|q''_l\rangle, \tag{26}$$

where we have used the invariance of the blocks to average $\hat{C}$ over all the $\boldsymbol{z'}$ coordinates. Eq. (26) correspond to Eq. (55) in the main text.